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Water-wave scattering by two submerged plane vertical barriers—Abel integral-equation approach

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Abstract The classical problem of surface water-wave scattering by two identical thin vertical barriers submerged in deep water and extending infinitely downwards from the same depth below the mean free surface, is reinvestigated here by an approach leading to the problem of solving a system of Abel integral equations. The reflection and transmission coefficients are obtained in terms of computable integrals. Known results for a single barrier are recovered as a limiting case as the separation distance between the two barriers tends to zero. The coefficients are depicted graphically in a number of figures which are identical with the corresponding figures given by Jarvis (J Inst Math Appl 7:207–215, 1971) who employed a completely different approach involving a Schwarz–Christoffel transformation of complex-variable theory to solve the problem.

Keywords Abel integral equations · Reflection and transmission coefficients · Two barriers · Wave scattering

1 Introduction

Problems of water-wave scattering by fixed thin plane vertical barriers in the linearised theory of water waves are being investigated in the literature for the last six decades by employing various mathematical techniques. The explicit solution of the problem of water-wave scattering by a fixed thin vertical barrier partially immersed in deep water was obtained by Ursell [1] by employing Havelock's expansion of the water-wave potential. He also considered the complementary problem when the barrier is submerged and extends infinitely downwards. For both cases the reflection and transmission coefficients were obtained in terms of modified Bessel functions. Evans [2] considered a plate submerged in deep water and employed complex-variable theory leading to solving a Riemann–Hilbert problem and obtained the reflection and transmission coefficients in terms of some definite integrals which can be computed numerically. Porter [3] investigated water-wave transmission through a submerged gap in a thin vertical wall using two approaches, one based on the use of Green's integral theorem and the other on complex-variable theory. Both approaches essentially lead to the problem of solving a Riemann–Hilbert problem. He also obtained

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the reflection and transmission coefficients in terms of some computable definite integrals. The uniqueness of the solution of this problem is assured by the uniqueness theorem involving two barriers given by Kuznetsov et al. [4] who considered uniqueness in water-wave problems containing two, three or four vertical barriers. Different analytical techniques have been developed in the literature to handle this class of problems. Of the various mathematical methods employed to solve such problems, the method involving a weakly singular integral equation of the Abel type appears to be elegant because of the simplicity of the structure of the kernel of the integral equation, as was demonstrated by Williams [5] while reinvestigating the problem of water-wave scattering by a thin vertical barrier partially immersed in deep water.

For two identical thin vertical barriers which are either partially immersed up to the same depth below the mean free surface or completely submerged from the same depth and extending infinitely downwards, the corresponding wave-scattering problems also admit of explicit solution. Levine and Rodemich [6] formulated the problem of two partially immersed barriers in terms of a reduced complex potential defined in the complex z-plane ($z = x + \mathrm{i}y$, y-axis being chosen along the vertical direction) and obtained the solution following a procedure which included the use of the Schwarz–Christoffel transformation of complex-variable theory. The complementary problem of two submerged fixed vertical barriers was investigated by Jarvis [7] following a similar procedure, and the reflection and transmission coefficients were obtained explicitly in terms of four definite integrals which could be evaluated numerically. However, the method employed by Levine and Rodemich [6] or by Jarvis [7] for two-barrier problems appears to be rather complicated. An alternative method based on solving an Abel integral equation is proposed here to tackle the two-barrier problem of Jarvis [7]. The present approach to the solution of an already investigated physical problem appears to be interesting enough, mathematically speaking, as it avoids a detailed analysis involving complex-function theory and conformal mapping, all of which are not easy to apply, in general, for problems of practical interest to engineers.

It may be noted that in the literature there exists a substantial amount of work involving two or more thin vertical barriers in deep as well as finite-depth water. In these, numerical estimates for the reflection and transmission coefficients have been obtained by employing some approximate analytical and numerical methods. Both normal and oblique incidence of a wave train have been considered. For example, Evans and Morries [8] derived an approximate expression for the reflection coefficient for the problem of two identical partially immersed barriers originally considered by Levine and Rodemich [6] by using a complementary variational approximation. Newman [9] also employed a matched asymptotic method to obtain approximate analytical result for the reflection coefficients when the two barriers are closely spaced. McIver [10] used matched eigenfunction expansions and an orthogonal expansion involving trigonometric functions to solve the problem of two partially immersed *unequal* vertical barriers. For water of finite depth and oblique incidence of the wave train, Mandal and Dolai [11], Das et al. [12] employed a Galerkin approximation to obtain fairly accurate numerical estimates for the reflection and transmission coefficients for a number of configurations involving two identical barriers. Porter and Evans [13] considered a number of wave-scattering problems involving thin vertical barriers with gaps and obtained very accurate numerical estimates for the reflection and transmission coefficients by using multi-term Galerkin approximations.

Here the two-barrier problem of Jarvis [7] is reinvestigated by an alternative method based on Fourier analysis for the expansion of the velocity-potential function, known as Havelock's expansion, leading to solving a system of Abel integral equations. A special theorem is proved here to solve the system of Abel integral equations. The reflection and transmission coefficients are then obtained in terms of expressions involving one modified Bessel function and two definite integrals which can be easily computed. These integrals appear to be somewhat simpler than the four definite integrals given by Jarvis [7]. Ursell's [1] explicit results for a single barrier, given in terms of modified Bessel functions, are derived from the results of two barriers, by making the separation length tend to zero. Taking the same set of values of different parameters as chosen by Jarvis [7], the reflection and transmission coefficients are depicted graphically against the wave number in a number of figures, and the curves in these figures are seen to coincide *exactly* with the corresponding curves given there. This demonstrates the correctness of the results obtained here.

In Sect. 2, we have presented the mathematical formulation of the boundary value problem and in Sect. 3, we have explained the detailed analysis leading to the special system of Able integral equations to be solved. These equations are then solved by utilizing a theorem, whose proof is presented in the Appendix. Section 4 is devoted to the presentation of the results involving the reflection and transmission coefficients and, in Sect. 5, certain interesting special cases will be examined along with a comparison with the known results of Jarvis [7].

2 Mathematical formulation of the problem

Let the two thin plane vertical barriers submerged in infinitely deep water occupy the positions $x = \pm a$, $y \geq h$; the y-axis is chosen vertically downwards into the fluid region with the plane $y = 0$ representing the mean position of the free surface. Assuming linear theory and two-dimensional irrotational motion, we describe a time-harmonic progressive wave train by the potential function $\Re\mathfrak{e}\{\phi_0(x, y)\mathrm{e}^{-\mathrm{i}\sigma t}\}$ with $\phi_0(x, y) = \mathrm{e}^{-ky+\mathrm{i}kx}$ from the direction of $x = -\infty$, is incident on the two barriers. Here σ is the angular frequency and $k = \sigma^2/g$ is the wave number, g denoting the acceleration due to gravity. Let $\phi(x, y)$ denote the complex-valued potential function describing the resulting motion in the fluid region. Then ϕ satisfies the boundary-value problem (BVP) described by

$$\nabla^2\phi = 0 \quad y \geq 0, \tag{2.1}$$

$$k\phi + \phi_y = 0 \quad \text{on } y = 0, \tag{2.2}$$

$$\phi_x = 0 \quad \text{on } x = \pm a,\ y \geq h, \tag{2.3}$$

$$r^{1/2}\nabla\phi = O(1) \quad \text{as } r = \{(x \mp a)^2 + y^2\}^{1/2} \to 0, \tag{2.4}$$

$$\nabla\phi \to 0 \quad \text{as } y \to \infty, \tag{2.5}$$

$$\phi(x, y) \sim \begin{cases} T\phi_0(x, y) & \text{as } x \to \infty, \\ \phi_0(x, y) + R\phi_0(-x, y) & \text{as } x \to -\infty, \end{cases} \tag{2.6}$$

where T and R, respectively, denote the unknown transmission and reflection coefficients, the determination of which is the prime concern here.

In the next section, the BVP described by (2.1) to (2.6) is reduced to a pair of coupled Abel-type singular integral equations.

3 Method of solution

The potential function $\phi(x, y)$ satisfying the BVP described by (2.1) to (2.6), has the following representation, after using Havelock's expansion of the water-wave potential (cf. [1]):

$$\phi(x, y) = \begin{cases} \mathrm{e}^{-ky}(\mathrm{e}^{\mathrm{i}kx} + R\mathrm{e}^{-\mathrm{i}kx}) + \dfrac{2}{\pi}\displaystyle\int_0^\infty A(\xi)L(\xi, y)\mathrm{e}^{\xi x}\mathrm{d}\xi, & x < -a, \\ \mathrm{e}^{-ky}(\alpha\mathrm{e}^{\mathrm{i}kx} + \beta\mathrm{e}^{-\mathrm{i}kx}) + \dfrac{2}{\pi}\displaystyle\int_0^\infty \{B(\xi)\mathrm{e}^{\xi x} + C(\xi)\mathrm{e}^{-\xi x}\}L(\xi, y)\mathrm{d}\xi, & -a < x < a, \\ T\mathrm{e}^{-ky+\mathrm{i}kx} + \dfrac{2}{\pi}\displaystyle\int_0^\infty D(\xi)L(\xi, y)\mathrm{e}^{-\xi x}\mathrm{d}\xi, & x > a, \end{cases} \tag{3.1}$$

where

$$L(\xi, y) = \xi\cos\xi y - k\sin\xi y; \tag{3.2}$$

α, β are unknown constants, $A(\xi)$, $B(\xi)$, $C(\xi)$ and $D(\xi)$ are unknown functions such that the integrals in (3.1) and in the mathematical analysis below in which they appear, are convergent.

Using the condition of continuity of ϕ_x across the lines $x = \pm a$, $y > 0$, we obtain two relations involving the unknown functions $A(\xi)$, $B(\xi)$, $C(\xi)$, $D(\xi)$ under the integral sign and the four unknown constants α, β, R, T. An application of Havelock's inversion theorem (cf. [1]) to each of these two relations gives rise to two equations, thus producing

$$\mathrm{e}^{-\mathrm{i}ka} - R\mathrm{e}^{\mathrm{i}ka} = \alpha\mathrm{e}^{-\mathrm{i}ka} - \beta\mathrm{e}^{\mathrm{i}ka}, \tag{3.3a}$$

$$T\mathrm{e}^{\mathrm{i}ka} = \alpha\mathrm{e}^{\mathrm{i}ka} - \beta\mathrm{e}^{-\mathrm{i}ka}, \tag{3.3b}$$

and

$$A(\xi) = B(\xi) - C(\xi)\mathrm{e}^{2\xi a}, \tag{3.4a}$$

$$D(\xi) = C(\xi) - B(\xi)\mathrm{e}^{2\xi a}. \tag{3.4b}$$

Again, continuity of ϕ across the gap $x = -a$, $0 < y < h$, produces

$$\frac{2}{\pi}\int_0^\infty \{A(\xi)\mathrm{e}^{-\xi a} - B(\xi)\mathrm{e}^{-\xi a} - C(\xi)\mathrm{e}^{\xi a}\}L(\xi, y)\mathrm{d}\xi = \mathrm{e}^{-ky-\mathrm{i}ka}\{\alpha + (\beta - R)\mathrm{e}^{2\mathrm{i}ka} - 1\}, \quad 0 < y < h.$$

Use of (3.3) and (3.4) simplifies this to

$$\frac{2}{\pi}\int_0^\infty C(\xi)\mathrm{e}^{\xi a}L(\xi, y)\mathrm{d}\xi = -(\beta - R)\mathrm{e}^{-ky-\mathrm{i}ka}, \quad 0 < y < h. \tag{3.5}$$

Similarly, continuity of ϕ across the gap $x = a$, $0 < y < h$, produces, after using (3.3) and (3.4):

$$\frac{2}{\pi}\int_0^\infty B(\xi)\mathrm{e}^{\xi a}L(\xi, y)\mathrm{d}\xi = -\beta\mathrm{e}^{-ky-\mathrm{i}ka}, \quad 0 < y < h. \tag{3.6}$$

From the condition that $\phi_x = 0$ on $x = \pm a$, $y > h$, we obtain

$$\frac{2}{\pi}\int_0^\infty \xi\{B(\xi)\mathrm{e}^{-\xi a} - C(\xi)\mathrm{e}^{\xi a}\}L(\xi, y)\mathrm{d}\xi = -\mathrm{i}k(\alpha\mathrm{e}^{-\mathrm{i}ka} - \beta\mathrm{e}^{\mathrm{i}ka})\mathrm{e}^{-ky}, \quad y > h, \tag{3.7}$$

and

$$\frac{2}{\pi}\int_0^\infty \xi\{B(\xi)\mathrm{e}^{\xi a} - C(\xi)\mathrm{e}^{-\xi a}\}L(\xi, y)\mathrm{d}\xi = -\mathrm{i}k(\alpha\mathrm{e}^{\mathrm{i}ka} - \beta\mathrm{e}^{-\mathrm{i}ka})\mathrm{e}^{-ky}, \quad y > h. \tag{3.8}$$

Now multiplying both sides of (3.5) to (3.8) by e^{-ky}, integrating with respect to y between 0 to $y(< h)$ in (3.5), (3.6) and between $y(> h)$ to ∞ in (3.7), (3.8), we obtain

$$\int_0^\infty C(\xi)\mathrm{e}^{\xi a}\sin\xi y\mathrm{d}\xi = -\frac{\pi}{2k}(\beta - R)\mathrm{e}^{\mathrm{i}ka}\sinh ky, \quad 0 < y < h, \tag{3.9}$$

$$\int_0^\infty B(\xi)\mathrm{e}^{\xi a}\sin\xi y\mathrm{d}\xi = -\frac{\pi}{2k}\beta\mathrm{e}^{-\mathrm{i}ka}\sinh ky, \quad 0 < y < h, \tag{3.10}$$

$$\int_0^\infty \xi\{B(\xi)\mathrm{e}^{-\xi a} - C(\xi)\mathrm{e}^{\xi a}\}\sin\xi y\mathrm{d}\xi = \frac{\mathrm{i}\pi}{4}(\alpha\mathrm{e}^{-\mathrm{i}ka} - \beta\mathrm{e}^{\mathrm{i}ka})\mathrm{e}^{-ky}, \quad y > h, \tag{3.11}$$

$$\int_0^\infty \xi\{B(\xi)\mathrm{e}^{\xi a} - C(\xi)\mathrm{e}^{-\xi a}\}\sin\xi y\mathrm{d}\xi = \frac{\mathrm{i}\pi}{4}(\alpha\mathrm{e}^{\mathrm{i}ka} - \beta\mathrm{e}^{-\mathrm{i}ka})\mathrm{e}^{-ky}, \quad y > h. \tag{3.12}$$

Let the left sides of (3.12) and (3.11) be equal to $g_1(y)$ and $g_2(y)$, respectively, for $0 < y < h$, where $g_1(y)$ and $g_2(y)$ are unknown functions. Then, by using the sine inversion formula, we find

$$\xi\{B(\xi)\mathrm{e}^{\xi a} - C(\xi)\mathrm{e}^{-\xi a}\} = \frac{2}{\pi}\int_0^h g_1(t)\sin\xi t\mathrm{d}t + \frac{\mathrm{i}}{2}(\alpha\mathrm{e}^{\mathrm{i}ka} - \beta\mathrm{e}^{-\mathrm{i}ka})\int_h^\infty \mathrm{e}^{-kt}\sin\xi t\mathrm{d}t, \tag{3.13}$$

and

$$\xi\{B(\xi)\mathrm{e}^{-\xi a} - C(\xi)\mathrm{e}^{\xi a}\} = \frac{2}{\pi}\int_0^h g_2(t)\sin\xi t\mathrm{d}t + \frac{\mathrm{i}}{2}(\alpha\mathrm{e}^{-\mathrm{i}ka} - \beta\mathrm{e}^{\mathrm{i}ka})\int_h^\infty \mathrm{e}^{-kt}\sin\xi t\mathrm{d}t. \tag{3.14}$$

Solving for $B(\xi)$, $C(\xi)$ from (3.13) and (3.14), and substituting these in (3.10) and (3.9), we obtain two coupled integral equations for $g_1(t)$ and $g_2(t)$ given by

$$\frac{1}{\pi}\int_0^h g_1(t)\mathcal{K}_1(t, y)\mathrm{d}t - \frac{1}{\pi}\int_0^h g_2(t)\mathcal{K}_2(t, y)\mathrm{d}t = f_1(y), \quad 0 < y < h, \tag{3.15}$$

$$\frac{1}{\pi}\int_0^h g_1(t)\mathcal{K}_2(t, y)\mathrm{d}t - \frac{1}{\pi}\int_0^h g_2(t)\mathcal{K}_1(t, y)\mathrm{d}t = f_2(y), \quad 0 < y < h, \tag{3.16}$$

where

$$(\mathcal{K}_1, \mathcal{K}_2)(t, y) = \int_0^\infty \frac{\sin \xi t \sin \xi y}{\xi \sinh 2\xi a}(\mathrm{e}^{2\xi a}, 1)\mathrm{d}\xi, \tag{3.17a}$$

so that

$$\mathcal{K}_1(t, y) = \frac{1}{2}\log\left|\frac{(y+t)\sinh\frac{\pi}{4a}(y+t)}{(y-t)\sinh\frac{\pi}{4a}(y-t)}\right|, \quad \mathcal{K}_2(t, y) = \frac{1}{2}\log\left|\frac{\cosh\frac{\pi}{4a}(y+t)}{\sinh\frac{\pi}{4a}(y-t)}\right|, \tag{3.17b}$$

$$f_1(y) = -\frac{\pi}{2k}\beta\mathrm{e}^{-\mathrm{i}ka}\sinh ky - \frac{\mathrm{i}}{4}(\alpha\mathrm{e}^{\mathrm{i}ka} - \beta\mathrm{e}^{-\mathrm{i}ka})\int_h^\infty \mathrm{e}^{-kt}\mathcal{K}_1(t, y)\mathrm{d}t$$
$$+\frac{\mathrm{i}}{4}(\alpha\mathrm{e}^{-\mathrm{i}ka} - \beta\mathrm{e}^{\mathrm{i}ka})\int_h^\infty \mathrm{e}^{-kt}\mathcal{K}_2(t, y)\mathrm{d}t, \quad 0 < y < h, \tag{3.18a}$$

$$f_2(y) = -\frac{\pi}{2k}(\beta - R)\mathrm{e}^{\mathrm{i}ka}\sinh ky - \frac{\mathrm{i}}{4}(\alpha\mathrm{e}^{\mathrm{i}ka} - \beta\mathrm{e}^{-\mathrm{i}ka})\int_h^\infty \mathrm{e}^{-kt}\mathcal{K}_2(t, y)\mathrm{d}t$$
$$+\frac{\mathrm{i}}{4}(\alpha\mathrm{e}^{-\mathrm{i}ka} - \beta\mathrm{e}^{\mathrm{i}ka})\int_h^\infty \mathrm{e}^{-kt}\mathcal{K}_1(t, y)\mathrm{d}t, \quad 0 < y < h. \tag{3.18b}$$

The coupled integral equations (3.15) and (3.16) are decoupled by addition and substraction, and the decoupled equations are given by

$$\frac{1}{\pi}\int_0^h g(t)\mathcal{K}(t, y)\mathrm{d}t = f(y), \quad 0 < y < h, \tag{3.19}$$

$$\frac{1}{\pi}\int_0^h G(t)\mathcal{L}(t, y)\mathrm{d}t = F(y), \quad 0 < y < h, \tag{3.20}$$

where

$$g(t) = g_1(t) - g_2(t), \quad G(t) = g_1(t) + g_2(t), \tag{3.21}$$

$$\mathcal{K}(t, y) = \mathcal{K}_1(t, y) + \mathcal{K}_2(t, y), \quad \mathcal{L}(t, y) = \mathcal{K}_1(t, y) - \mathcal{K}_2(t, y), \tag{3.22}$$

$$f(y) = f_1(y) + f_2(y), \quad F(y) = f_1(y) - f_2(y), \tag{3.23}$$

It is important to note that

$$f(0) = 0, \quad F(0) = 0.$$

It follows from (3.17b) that the kernels $\mathcal{K}(t, y)$ and $\mathcal{L}(t, y)$ are given by

$$\mathcal{K}(t, y) = \frac{1}{2}\log\left|\frac{y+t}{y-t}\right| + \frac{1}{2}\log\left|\frac{\tanh\frac{\pi y}{2a} + \tanh\frac{\pi t}{2a}}{\tanh\frac{\pi y}{2a} - \tanh\frac{\pi t}{2a}}\right|,$$
$$\mathcal{L}(t, y) = \frac{1}{2}\log\left|\frac{y+t}{y-t}\right| + \frac{1}{2}\log\left|\frac{\sinh\frac{\pi y}{2a} + \sinh\frac{\pi t}{2a}}{\sinh\frac{\pi y}{2a} - \sinh\frac{\pi t}{2a}}\right|. \tag{3.24}$$

Using the result (proved by elementary integration)

$$\frac{1}{2}\log\left|\frac{Y+T}{Y-T}\right| = \int_0^{\min(Y,T)} \frac{X}{[(Y^2 - X^2)(T^2 - X^2)]^{1/2}}\mathrm{d}X,$$

we may easily show that, if $\psi(y)$ is an increasing function, then

$$\frac{1}{2}\log\left|\frac{\psi(y)+\psi(t)}{\psi(y)-\psi(t)}\right| = \int_0^{\min(y,t)} \frac{\psi'(\eta)\psi(\eta)}{[\{\psi^2(y) - \psi^2(\eta)\}\{\psi^2(t) - \psi^2(\eta)\}]^{1/2}}\mathrm{d}\eta. \tag{3.25}$$

Using the result (3.25) in the logarithmic expressions for the kernel $\mathcal{K}(t, y)$, we find that the integral equation (3.19) is equivalent to the Abel-type integral equation

$$\int_0^y \frac{\eta p(\eta)}{(y^2 - \eta^2)^{1/2}}\mathrm{d}\eta + \frac{\pi}{2a}\int_0^y \frac{q(\eta)\mathrm{sech}^2\frac{\pi\eta}{2a}\tanh\frac{\pi\eta}{2a}}{\left(\tanh^2\frac{\pi y}{2a} - \tanh^2\frac{\pi\eta}{2a}\right)^{1/2}}\mathrm{d}\eta = f(y), \quad 0 < y < h, \tag{3.26}$$

where

$$p(\eta) = \int_\eta^h \frac{g(t)}{(t^2-\eta^2)^{1/2}}\mathrm{d}t, \quad q(\eta) = \int_\eta^h \frac{g(t)}{\left(\tanh^2\frac{\pi t}{2a} - \tanh^2\frac{\pi\eta}{2a}\right)^{1/2}}\mathrm{d}t, \quad 0<\eta<h, \tag{3.27}$$

so that $p(h)=0, q(h)=0$ for consistency. The functions $p(\eta), q(\eta)$ are not independent, and they satisfy the relation

$$\int_y^h \frac{\eta p(\eta)}{(\eta^2-y^2)^{1/2}}\mathrm{d}\eta - \frac{\pi}{2a}\int_y^h \frac{q(\eta)\mathrm{sech}^2\frac{\pi\eta}{2a}\tanh\frac{\pi\eta}{2a}}{\left(\tanh^2\frac{\pi\eta}{2a} - \tanh^2\frac{\pi y}{2a}\right)^{1/2}}\mathrm{d}\eta = 0, \quad 0<y<h, \tag{3.28}$$

obtained by equating the two forms of $g(y)$ found by inverting the relations in (3.27).

Similarly, the integral equation (3.20) is equivalent to the Abel-type integral equation

$$\int_0^y \frac{\eta P(\eta)}{(y^2-\eta^2)^{1/2}}\mathrm{d}\eta + \frac{\pi}{2a}\int_0^y \frac{Q(\eta)\cosh\frac{\pi\eta}{2a}\sinh\frac{\pi\eta}{2a}}{\left(\sinh^2\frac{\pi y}{2a} - \sinh^2\frac{\pi\eta}{2a}\right)^{1/2}}\mathrm{d}\eta = F(y), \quad 0<y<h, \tag{3.29}$$

where

$$P(\eta) = \int_\eta^h \frac{G(t)}{(t^2-\eta^2)^{1/2}}\mathrm{d}t, \quad Q(\eta) = \int_\eta^h \frac{G(t)}{\left(\sinh^2\frac{\pi t}{2a} - \sinh^2\frac{\pi\eta}{2a}\right)^{1/2}}\mathrm{d}t, \quad 0<\eta<h, \tag{3.30}$$

so that $P(h)=0, Q(h)=0$ for consistency. As in the case of $p(\eta)$ and $q(\eta)$, here also, $P(\eta)$ and $Q(\eta)$ are not independent, and they satisfy the relation

$$\int_y^h \frac{\eta P(\eta)}{(\eta^2-y^2)^{1/2}}\mathrm{d}\eta - \frac{\pi}{2a}\int_y^h \frac{Q(\eta)\cosh\frac{\pi\eta}{2a}\sinh\frac{\pi\eta}{2a}}{\left(\sinh^2\frac{\pi\eta}{2a} - \sinh^2\frac{\pi y}{2a}\right)^{1/2}}\mathrm{d}\eta = 0, \quad 0<y<h. \tag{3.31}$$

Solutions for $p(\eta)$, $P(\eta)$ (and hence $q(\eta)$, $Q(\eta)$) can be obtained from the following theorem.

Theorem

Let

$$\int_x^h \frac{\psi_1'(t)p(t)}{(\psi_1(t)-\psi_1(x))^{1/2}}\mathrm{d}t = \int_x^h \frac{\psi_2'(t)q(t)}{(\psi_2(t)-\psi_2(x))^{1/2}}\mathrm{d}t, \quad 0<x<h \tag{3.32}$$

where $p(h)=0, q(h)=0$; $\psi_1(t), \psi_2(t)$ *are monotonically increasing functions in* $(0,h)$; $\psi_1(0)=0$; $\psi_2(0)=0$; $\psi_1(t)$ *and* $\psi_2(t)$ *are even functions of* t.
Then

$$\int_0^x \frac{\psi_1'(t)p(t)}{(\psi_1(x)-\psi_1(t))^{1/2}}\mathrm{d}t = \int_0^x \frac{\psi_2'(t)q(t)}{(\psi_2(x)-\psi_2(t))^{1/2}}\mathrm{d}t, \quad 0<x<h \tag{3.33}$$

The proof of this theorem is given in the Appendix.

By using this theorem with $\psi_1(t)=t^2, \psi_2(t)=\tanh^2\frac{\pi t}{2a}$, it is found that the two Abel integrals on the left side of (3.26) are same, and thus (3.26) becomes

$$\int_0^x \frac{\eta p(\eta)}{(x^2-\eta^2)^{1/2}}\mathrm{d}\eta = \frac{1}{2}f(x), \quad 0<x<h. \tag{3.34}$$

By Abel inversion, Eq. 3.34 produces

$$\eta p(\eta) = \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\eta}\int_0^\eta \frac{xf(x)}{(\eta^2-x^2)^{1/2}}\mathrm{d}x, \quad 0<\eta<h$$

so that

$$p(\eta) = \frac{1}{2\pi}\int_0^\eta \frac{f'(x)}{(\eta^2-x^2)^{1/2}}\mathrm{d}x, \quad 0<\eta<h. \tag{3.35}$$

Since $p(h) = 0$, we find that

$$\int_0^h \frac{f'(x)}{(h^2-x^2)^{1/2}}\mathrm{d}x = 0. \tag{3.36}$$

Adopting a similar procedure for the Abel integral equation (3.29) (where $\psi_1(t) = t^2$, $\psi_2(t) = \sinh^2 \frac{\pi t}{2a}$), we obtain the result

$$\int_0^h \frac{F'(x)}{(h^2-x^2)^{1/2}}\mathrm{d}x = 0. \tag{3.37}$$

The Eqs. 3.36, 3.37 together with (3.3a), (3.3b) will produce the four unknown constants α, β, R, T explicitly. This is carried out in the next section.

4 Reflection and transmission coefficients

The functions $f(x)$ and $F(x)$ are defined by (3.23) where $f_1(x)$ and $f_2(x)$ are given in (3.18). Substituting the appropriate expressions for $f(x)$ and $F(x)$ in (3.36) and (3.37), carrying out the necessary integrations, and substituting for α, β in terms of R, T from (3.3), we finally obtain

$$R+T = \frac{\mathrm{i}\left\{\frac{\pi^2}{2}I_0(kh) - \frac{U}{2}\sin 2ka\right\} - U\sin^2 ka}{\mathrm{i}\left\{\frac{\pi^2}{2}I_0(kh) - \frac{U}{2}\sin 2ka\right\} + U\sin^2 ka} \tag{4.1}$$

and

$$R-T = -\frac{\left\{\frac{\pi^2}{2}I_0(kh) + \frac{V}{2}\sin 2ka\right\} + \mathrm{i}V\cos^2 ka}{\left\{\frac{\pi^2}{2}I_0(kh) + \frac{V}{2}\sin 2ka\right\} - \mathrm{i}V\cos^2 ka}, \tag{4.2}$$

where

$$\begin{aligned} U, V &= \int_0^h \frac{\mathrm{d}y}{(h^2-y^2)^{1/2}}\left[\int_h^\infty \mathrm{e}^{-kt} \lim_{\epsilon\to 0+}\left\{\int_0^\infty \frac{\mathrm{e}^{-\epsilon\xi}\sin\xi t\cos\xi y}{\sinh 2\xi a}(1\pm \mathrm{e}^{2\xi a})\mathrm{d}\xi\right\}\mathrm{d}t\right] \\ &= \mathrm{e}^{-kh}\int_0^\infty \frac{(k\sin\xi h + \xi\cos\xi h)J_0(\xi h)}{\xi^2+k^2}\frac{(1\pm \mathrm{e}^{2\xi a})}{\sinh 2\xi a}\mathrm{d}\xi; \end{aligned} \tag{4.3}$$

here $I_0(kh)$ is the modified Bessel function and $J_0(\xi h)$ is the Bessel function. Thus R and T are obtained as

$$R, T = \frac{1}{2}\left[\frac{\mathrm{i}\left\{\frac{\pi^2}{2}I_0(kh) - \frac{U}{2}\sin 2ka\right\} - U\sin^2 ka}{\mathrm{i}\left\{\frac{\pi^2}{2}I_0(kh) - \frac{U}{2}\sin 2ka\right\} + U\sin^2 ka} \mp \frac{\left\{\frac{\pi^2}{2}I_0(kh) + \frac{V}{2}\sin 2ka\right\} + \mathrm{i}V\cos^2 ka}{\left\{\frac{\pi^2}{2}I_0(kh) + \frac{V}{2}\sin 2ka\right\} - \mathrm{i}V\cos^2 ka}\right]. \tag{4.4}$$

It is straightforward to observe that R, T satisfy

$$|R|^2 + |T|^2 = 1,$$

which is the energy identity.

As observed by Jarvis [7], there is an infinite sequence of values of kh for which R vanishes. These are determined by (using the same notation of Jarvis [7]),

$$\lambda\tan^2 ka + 2\lambda\mu\tan ka - \mu = 0, \tag{4.5}$$

where

$$\lambda = \frac{U}{\pi^2 I_0(kh)}, \quad \mu = \frac{V}{\pi^2 I_0(kh)}. \tag{4.6}$$

The roots of (4.5) have the same form as those given by Jarvis [7] since the form of (4.5) is exactly same but with apparently different forms for λ and μ.

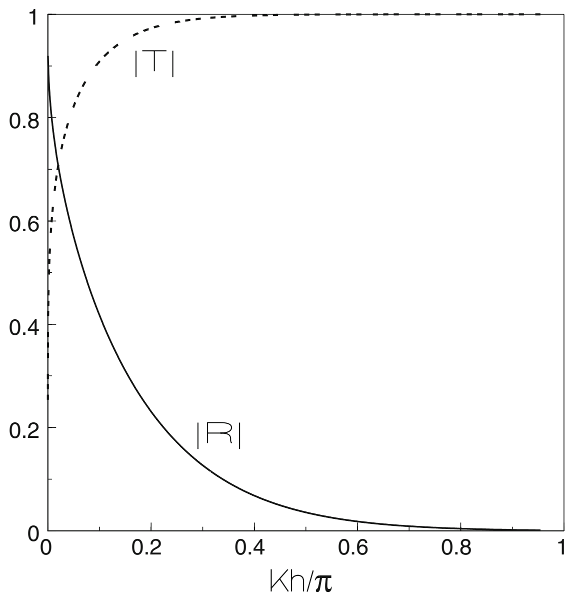

Fig. 1 $|R|$, $|T|$ for $a/h = .01$

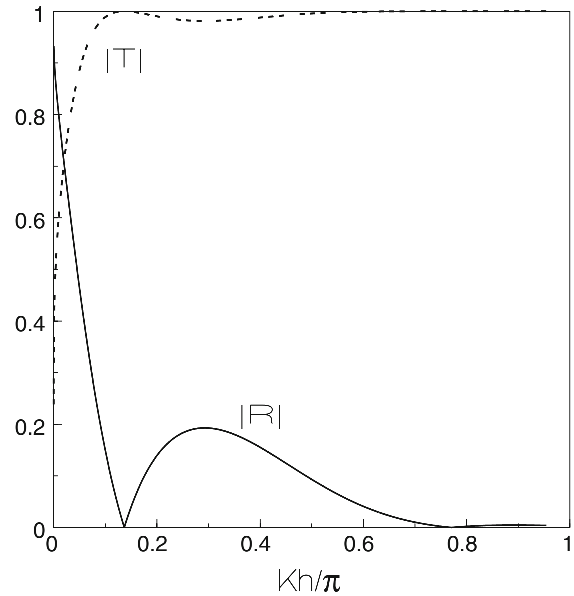

Fig. 2 $|R|$, $|T|$ for $a/h = 1.0$

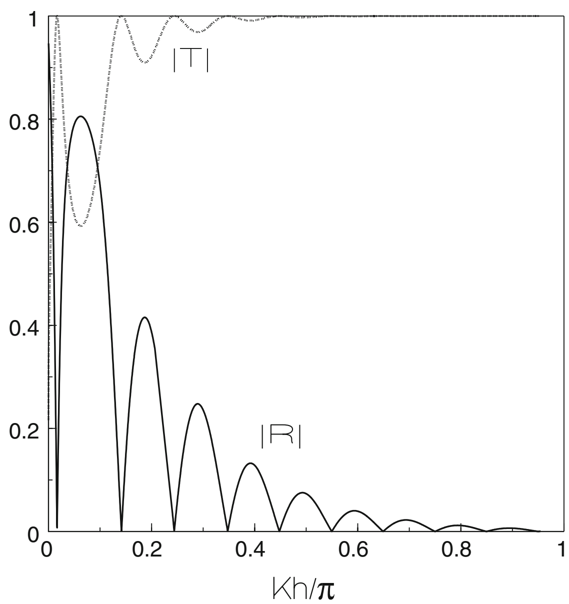

Fig. 3 $|R|$, $|T|$ for $a/h = 5.0$

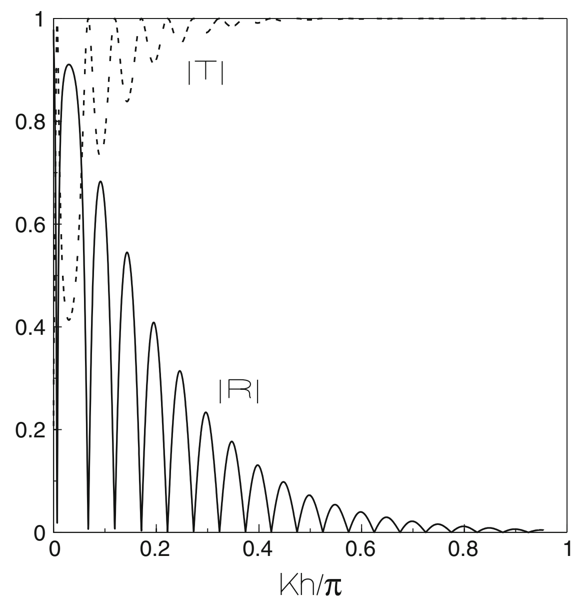

Fig. 4 $|R|$, $|T|$ for $a/h = 10.0$

5 Discussion

5.1 Approximation of R, T for small separation length

It is possible to derive the results for a single submerged barrier from (4.4) by making $a \to 0$ but keeping kh fixed. As $a \to 0$,

$$U \sin ka \to U_0, \quad V \to \frac{\pi}{2} K_0(kh),$$

where

$$U_0 = k\mathrm{e}^{-kh} \int_0^\infty \frac{(k \sin \xi h + \xi \cos \xi h)}{\xi(\xi^2 + k^2)} J_0(\xi h) \mathrm{d}\xi.$$

Using these results in (4.4), we find that as $a \to 0$

$$R \to R_0, \quad T \to T_0,$$

where

$$R_0 = \frac{K_0(kh)}{K_0(kh) + \mathrm{i}\pi I_0(kh)}, \quad T_0 = \frac{\mathrm{i}\pi I_0(kh)}{K_0(kh) + \mathrm{i}\pi I_0(kh)}.$$

These are well-known results obtained long ago by Ursell [1].

5.2 Numerical results

The functions $|R|$, $|T|$ given in (4.4) are depicted graphically against kh/π for different values of a/h in a number of figures to compare our results with those given in Jarvis [7]. In Fig. 1, $|R|$, $|T|$ are shown for small separation length($a/h = 0.1$). The curves in this figure correspond to the reflection and transmission coefficients $|R_0|$ and $|T_0|$, respectively, for a single barrier. As in [7], the difference in the result for $a/h = 0$ and $a/h = 0.1$ is not appreciable.

In Figs. 2–4, $|R|$ and $|T|$ are shown against kh/π for $a/h = 1, 5, 10$. It is observed that all the curves in the Figs. 1–4 coincide exactly with the corresponding curves in Figs. 1–4 of [7]. This shows the correctness of the results for the reflection and transmission coefficients obtained here.

6 Conclusion

Problems of the scattering of surface water waves by vertical barriers have a long history, starting from the work of Dean [14] and Ursell [1]. A range of excellent mathematical ideas have been developed by a large number of workers (cf. [2,3,5,15] and others). Interesting methods of solution of this class of mixed boundary-value problems, leading to solutions of integral equations, have been discovered and demonstrated to handle such problems. Several generalizations of the single vertical barrier problem of Dean [14] and Ursell [1] have also been examined in the literature.

The generalization, involving a pair of thin vertical barriers submerged in deep water from the same depth below the mean free surface was handled for its complete solution by Jarvis [7], by the aid of complex-function theory involving the Schwarz–Christoffel transformation. Levine and Rodemitch [6] mentioned earlier a similar method to handle the problem of scattering of surface water waves by any number of parallelly placed vertical barriers and explained it for the case of a pair of thin vertical barriers partially immersed in deep water up to the same depth.

Of the various analytical methods of solution utilized to handle this class of boundary-value problems, the method of Williams [5] appears to be simple and straightforward; it uses the simplest singular integral equation of the first kind, with a weak singularity, which is known as the Abel integral equation. In the present paper we have explained the use of a system of two Abel integral equations to handle the problem of the scattering of surface water waves by two parallelly placed surface-piercing vertical barriers of the same height. The method of reducing of the main boundary-value problem to a system of two Abel integral equations involves the use of Fourier analysis and Havelock's expansion theorem, in a straightforward manner. Then, the resulting system of Abel integral equations is handled for their complete solution. The determination of the reflection and transmission coefficients has been completed in terms of computable integrals. The results for the particular case of a single barrier is fully recovered, by utilizing a standard limiting procedure, and this serves as a check of the method employed to handle the general problem considered here.

The complementary problem of two parallelly placed immersed vertical barriers has also been handled successfully, by the present method, involving a system of two Abel integral equations and this will be the subject of a separate publication. The case of two vertical plates submerged in deep water can perhaps be handled by this method. Also, it will be interesting to investigate the situation when any number of vertical barriers are present as mentioned by Levine and Rodemich [6], by this method.

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Appendix, Proof of the Theorem

Let $S(x)$ denote the right side of (3.32). Then $S(h) = 0$. Now by Abel inversion, Eq. 3.32 gives, after some calculations

$$p(t) = -\frac{1}{\pi}\int_t^h \frac{S'(u)}{(\psi_1(u)-\psi_1(t))^{1/2}}\,\mathrm{d}u, \quad 0 < t < h. \tag{A1}$$

Using (A1), we obtain the left side of (3.33) as

$$\begin{aligned} L &= -\frac{1}{\pi}\int_0^x \frac{\psi_1'(t)}{(\psi_1(x)-\psi_1(t))^{1/2}}\left\{\left(\int_t^x + \int_x^h\right)\frac{S'(u)}{(\psi_1(u)-\psi_1(t))^{1/2}}\mathrm{d}u\right\}\mathrm{d}t \\ &= -\frac{1}{\pi}\left[\int_0^x S'(u)\left\{\int_0^u \frac{\psi_1'(t)}{(\psi_1(x)-\psi_1(t))^{1/2}(\psi_1(u)-\psi_1(t))^{1/2}}\mathrm{d}t\right\}\mathrm{d}u\right. \\ &\quad \left.+\int_x^h S'(u)\left\{\int_0^x \frac{\psi_1'(t)}{(\psi_1(x)-\psi_1(t))^{1/2}(\psi_1(u)-\psi_1(t))^{1/2}}\mathrm{d}t\right\}\mathrm{d}u\right]. \end{aligned}$$

Now

$$\begin{aligned} &\int_0^{\min(u,x)} \frac{\psi_1'(t)}{(\psi_1(x)-\psi_1(t))^{1/2}(\psi_1(u)-\psi_1(t))^{1/2}}\mathrm{d}t \\ &\quad = \log\left|\frac{\sqrt{\psi_1(x)}+\sqrt{\psi_1(u)}}{\sqrt{\psi_1(x)}-\sqrt{\psi_1(u)}}\right|. \end{aligned}$$

Thus

$$\begin{aligned} L &= \frac{1}{\pi}\int_0^h S'(u)\log\left|\frac{\sqrt{\psi_1(x)}+\sqrt{\psi_1(u)}}{\sqrt{\psi_1(x)}-\sqrt{\psi_1(u)}}\right|\mathrm{d}u \\ &= \frac{1}{\pi}\,⨍_0^h S(u)\frac{\psi_1'(u)}{\sqrt{\psi_1(u)}}\frac{\sqrt{\psi_1(x)}}{(\psi_1(x)-\psi_1(u))}\mathrm{d}x, \end{aligned}$$

where the integral is in the sense of a Cauchy principal value. Substituting the expression for $S(u)$, we find

$$L = \frac{\sqrt{\psi_1(x)}}{\pi}\int_0^h\left(\int_0^t \frac{\psi_1'(u)}{\sqrt{\psi_1(u)}}\frac{\mathrm{d}u}{(\psi_1(x)-\psi_1(u))(\psi_2(t)-\psi_2(u))^{1/2}}\right)\psi_2'(t)q(t)\mathrm{d}t$$

or, after changing the order of integration,

$$= \frac{\sqrt{\psi_1(x)}}{\pi}\left(\int_0^x + \int_x^h\right)\psi_2'(t)q(t)\left[\int_0^t \frac{\psi_1'(u)}{\sqrt{\psi_1(u)}}\frac{\mathrm{d}u}{(\psi_1(x)-\psi_1(u))(\psi_2(t)-\psi_2(u))^{1/2}}\right]\psi_2'(t)q(t)\mathrm{d}t. \tag{A2}$$

Thus we have to evaluate the inner integral

$$I = \int_0^t \frac{\psi_1'(u)}{\sqrt{\psi_1(u)}}\frac{\mathrm{d}u}{(\psi_1(x)-\psi_1(u))(\psi_2(t)-\psi_2(u))^{1/2}} \tag{A3}$$

for $t < x$ and $t > x$ (for $t > x$, the integral is in the sense of CPV).

Now in (A3) we put $\psi_1(u) = \theta$, $\psi_1(t) = \eta$, and $\psi_1(x) = \zeta$. Since $\psi_1(0) = 0$ and $\psi_1(x)$ is monotonically increasing in $(0, h)$, we get $\eta > \zeta$ for $t > x$ and $\eta < \zeta$ for $t < x$. If we also put

$\psi_2(t) = \psi_2(\psi_1^{-1}(\eta)) \equiv l(\eta)$, say,

then, as $\psi_2(0) = 0, \psi_1^{-1}(0) = 0, l(0) = 0$ and $l(\eta)$ is monotonically, increasing for $t \in (0, h)$ (i.e., $\eta \in (0, \psi_1^{-1}(h))$). Hence

$$I = \int_0^{\eta} \frac{1}{\sqrt{\theta}(l(\eta) - l(\theta))^{1/2}} \frac{\mathrm{d}\theta}{(\zeta - \theta)},$$

where, for $t < x$, i.e., for $\eta > \zeta$, the integral is in the sense of CPV.

Now, let

$$I(\zeta) = \int_0^{\eta} \frac{1}{\sqrt{x}(l(\eta) - l(x))^{1/2}} \frac{\mathrm{d}x}{\zeta - x}.$$

We introduce two complex variables: $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$, with $u = l(x)$ (for $x > 0$) and v is a function of y such that $v(0) = 0$. Exact forms of w, u and v are not needed for our purpose, as described below. Consider the function

$$F(z, w) = \frac{1}{\sqrt{z(w - l(\eta))}}$$

and the integral

$$J(\alpha) = \int_{\Gamma} \frac{F(z, w)}{z - \alpha} \mathrm{d}z, \tag{A4}$$

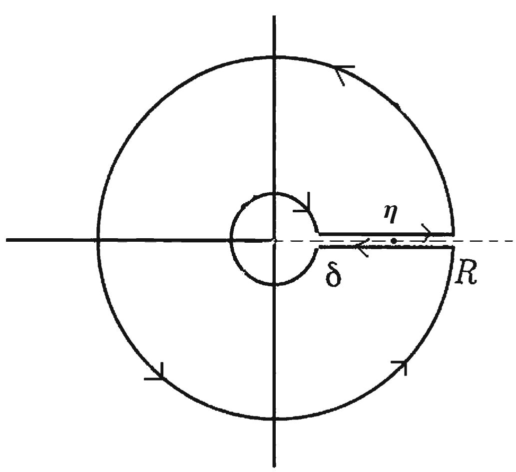
where $\alpha = \zeta + \mathrm{i}\beta$ is a fixed complex number in the complex z-plane cut along the real line from 0 to infinity and Γ is a closed contour in z-plane, as shown in Fig. 5.

Now utilizing the fact that the function $\sqrt{w - l(\eta)}$ is sectionally analytic in the complex w-plane, cut along the line joining the point $u = l(\eta)$ to infinity along the u-axis and the function $\sqrt{z}$ is sectionally analytic in the z-plane, cut along the positive real x-axis, and the following limiting values

(i) $\sqrt{z} \to \pm\sqrt{x}$ as $y \to 0\pm$ for $x > 0$,

(ii) $\sqrt{w - l(\eta)} \to \mathrm{i}\sqrt{l(\eta) - u}$ as $v \to 0\pm$ for $0 < u < l(\eta)$, i.e., for $0 < x < \eta$,
and
$\sqrt{w - l(\eta)} \to \pm\sqrt{u - l(\eta)}$ as $v \to 0\pm$ for $u > l(\eta)$, i.e., for $x > \eta$,

Fig. 5 Z-plane



hold, we obtain

$$J(\alpha)=2\mathrm{i}\int_0^{\eta}\frac{\mathrm{d}x}{(\alpha-x)\sqrt{x(l(\eta)-l(x))}}=2\mathrm{i}I(\alpha). \tag{A5}$$

However, by using the Cauchy's residue theorem, we find that

$$J(\alpha)=\frac{2\pi\mathrm{i}}{\sqrt{\alpha(w-l(\eta))}}. \tag{A6}$$

Thus (A5) and (A6) give

$$I(\alpha)=\frac{\pi}{\sqrt{\alpha(w-l(\eta))}}.$$

Now defining

$$I^{\pm}(\zeta)=\lim_{\beta\to\pm 0} I(\alpha)(\alpha>0),$$

we obtain

$$I^{+}(\zeta)=\begin{cases}\dfrac{\pi}{\mathrm{i}\sqrt{\zeta(l(\eta)-l(\zeta))}} & \text{for } 0<\zeta<\eta,\\ \dfrac{\pi}{\sqrt{\zeta(l(\zeta)-l(\eta))}} & \text{for } \zeta>\eta,\end{cases} \tag{A7}$$

$$I^{-}(\zeta)=\begin{cases}\dfrac{\pi}{-\mathrm{i}\sqrt{\zeta(l(\eta)-l(\zeta))}} & \text{for } 0<\zeta<\eta,\\ \dfrac{\pi}{\sqrt{\zeta(l(\zeta)-l(\eta))}} & \text{for } \zeta>\eta.\end{cases} \tag{A8}$$

Use of the Plemelj formulae produces

$$I(\zeta)=\frac{1}{2}(I^{+}(\zeta)+I^{-}(\zeta))=\begin{cases}0 & \text{for } 0<\zeta<\eta,\\ \dfrac{\pi}{\sqrt{\zeta(l(\zeta)-l(\eta))}} & \text{for } \zeta>\eta.\end{cases} \tag{A9}$$

Thus we obtain the following result

$$\int_0^t\frac{\psi_1'(u)}{\sqrt{\psi_1(u)}}\frac{\mathrm{d}u}{(\psi_1(x)-\psi_1(u))(\psi_2(t)-\psi_2(u))^{1/2}}=\begin{cases}0 & \text{for } t>x,\\ \dfrac{\pi}{\sqrt{\psi_1(x)(\psi_2(x)-\psi_2(t))}} & \text{for } t<x,\end{cases} \tag{A10}$$

so that

$$L=\int_0^x\frac{\psi_2'(t)q(t)\mathrm{d}t}{\sqrt{\psi_2(x)-\psi_2(t)}}.$$

Thus the relation (3.33) has been proved.

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